A NOTE ON MINIMAL TOPOLOGICAL SPACES

BY

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ABSTRACT

Let (X,τ) be a completely Hausdorff space. Let P be any topological property which is implied by complete regularity. Let (X,τ) be minimal-P. Then it has been shown that (X,τ) is completely regular and hence compact.

Let (X,τ) be a topological space. Let $P(X)$ be the set of all topologies u which can be defined on X such that (X,u) has the property P, where P may denote any topological property. The space (X,τ) is said to be minimal-P for any topological property P if τ is a minimal element in the set $P(X)$. A space (X,τ) is said to be P-closed if it is closed in every space Y in which it can be embedded, where the space Y has the property P. P-closed spaces for $P=$ Hausdorff are known to be H-closed. In the present note, we have obtained some theorems of the general type concerning minimal-P and P-closed spaces. Using these theorems several new results can be obtained. Several known results follow as corollaries. For $P=$ completely Hausdorff, our Theorem 1 is mentioned and Theorem 2 has been proved in [2]. The proof of Theorem 2 is essentially based on the idea of the proof of the corresponding theorem in [2].

We shall assume the Hausdorff property with every separation axiom here.

THEOREM 1. *A space (X,τ) is minimal P if and only if every one-to-one continuous function onto a space with property P is a homeomorphism, where P stands for any topological property.*

PROOF. Let (X,τ) be a minimal-P space. Let f be a one-to-one continuous function from (X,τ) onto a space (Y,u) with property P. Let $\tau^* = \{G: f(G) \in u\}$. Since f is one-to-one, it is easy to verify that τ^* is a topology for X and $f: (X,\tau^*) \to (Y,u)$ is open. Note that for each $U \in u$, $f[f^{-1}(U)] = U$. Therefore, $f^{-1}(U)$

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$\in \tau^*$ for each $U \in u$. Thus f: $(X, \tau^*) \to (Y, u)$ is continuous and hence is a homeomorphism. Since f is continuous, $\tau^* \subseteq \tau$, for if $G \in \tau^*$, then $f(G) \in u$ and therefore $G = f^{-1}(f(G)) \in \tau$. Now (X, τ^*) is a space with the property P. Since $\tau^* \subseteq \tau$ and (X, τ) is minimal-P, we have $\tau = \tau^*$. Hence f is a homeomorphism.

Conversely, suppose, if possible, that there exists a topology τ^* weaker than τ such that (X, τ^*) has the property P. Then the identity map from (X, τ) onto (X, τ^*) is a one-to-one continuous function and hence a homeomorphism. Thus $\tau = \tau^*$ and (X, τ) is minimal-P.

DEFINITION. A space (X, τ) is said to be *completely Hausdorff* if for every pair of points x_1 and x_2 in X, there exists a continuous function f from (X, τ) onto the closed unit interval $[0, 1]$ such that $f(x_1) \neq f(x_2)$.

THEOREM 2. *Let (X, τ) be a completely Hausdorff space. Let P be any topological property which is implied by complete regularity. If (X, τ) is minimal-P, then (X, τ) is completely-regular and hence compact.*

PROOF. Let F be the set of all continuous functions from (X, τ) onto $[0, 1]$. Let $[0, 1]^F$ denote the product of F copies of $[0,1]$. Define $g: (X, \tau) \to [0, 1]^F$ by $(g(x))_f = f(x)$. Let each projection mapping be denoted by p_f. Then $p_f o g = f$ for all $f \in F$ and hence $p_f o g$ is continuous for each $f \in F$ and therefore g is continuous. Also the function g is one-one because if $x_1 \neq x_2$, then there exists an $f \in F$ such that $f(x_1) \neq f(x_2)$ and hence $g(x_1) \neq g(x_2)$. Now we have a function $g: X \to g(X)$, $g(X) \subseteq [0, 1]^F$ which is one-one continuous function from X onto $g(X)$, where, being completely-regular, $g(X)$ has the property P. Since (X, τ) is minimal-P, g is a homeomorphism in view of Theorem 1 and hence (X, τ) is completely-regular. Since every minimal completely-regular space is compact (cf [1]), (X, τ) is compact.

THEOREM 3. *Let P stand for any property which implies complete-regularity and which is possessed by compact Hausdorff spaces. Then every space X is P-closed if and only if X is compact Hausdorff.*

PROOF. Let X be a space and P be a property which implies complete-regularity and is possessed by compact Hausdorff spaces. Let X be P-closed. Then X should be a closed subset in its Stone-Čech compactification βX. Hence $X = \beta X$ and is therefore compact.

The converse is obvious in view of the fact that compact Hausdorff spaces are H-closed.

References

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